

Modified Stancu-type Dunkl generalization of Szász-Kantorovich-operators

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Abstract

In this paper, we introduce a modification of the Szász-Mirakjan-Kantorovich operators as well as Stancu operators [9] (or a Dunkl generalization of modified Szász-Mirakjan-Kantorovich operators [5]) which preserve the linear functions. These types of operators modification enables better error estimation on the interval $[\frac{1}{2}, \infty)$ than the classical Dunkl Szász-Mirakjan-Kantorovich as well as Stancu operators. We obtain some approximation results via well known Korovkin's type theorem, weighted Korovkin's type theorem convergence properties by using the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class.

Keywords and phrases: Dunkl analogue; generating functions; generalization of exponential function; Szász operator; Korovkin type Theorem; modulus of continuity; weighted modulus of continuity.

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1. INTRODUCTION AND PRELIMINARIES

In 1912, S.N Bernstein [3] introduced the following sequence of operators $B_n : C[0, 1] \rightarrow C[0, 1]$ defined by

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1]. \quad (1.1)$$

for $n \in \mathbb{N}$ and $f \in C[0, 1]$.

In 1930 the first Bernstein-Kantorovich operators [10]. And in 1950 Szász operators [18] for $x \geq 0$, defined as follows:

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad f \in C[0, \infty). \quad (1.2)$$

In recent years, many results about the generalization of Szász operators have been obtained by several mathematicians [1, 2, 11, 12, 13, 15, 17]. Recently, Sucu [16] define a Dunkl analogue of Szász operators via a generalization of the exponential function given by [14] as follows:

$$S_n^*(f; x) := \frac{1}{e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_{\mu}(k)} f\left(\frac{k + 2\mu\theta_k}{n}\right), \quad (1.3)$$

where $x \geq 0$, $f \in C[0, \infty)$, $\mu \geq 0$, $n \in \mathbb{N}$
and

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)}. \quad (1.4)$$

Here

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma(k + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})}, \quad \gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma(k + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})}. \quad (1.5)$$

There is given a recursion for γ_μ

$$\gamma_\mu(k+1) = (k+1 + 2\mu\theta_{k+1})\gamma_\mu(k), \quad k = 0, 1, 2, \dots,$$

where

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N} + 1. \end{cases}$$

For $\mu \geq 0$, $x \geq 0$, and $f \in C[0, \infty)$, Gürhan İçöz [9] gave a Dunkl generalization of Kantorovich type integral generalization of Szász operators as follows:

$$T_n^*(f; x) = \frac{n}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} f\left(\frac{nt + \alpha}{n + \beta}\right) dt, \quad (1.6)$$

where $e_\mu(x)$ and γ_k are defined in [16] by (1.4), (1.5).

Lemma 1.1. (1) $T_n^*(1; x) = 1$,
(2) $T_n^*(t; x) = \frac{n}{n+\beta}x + \frac{1}{n+\beta}(\alpha + \frac{1}{2})$,
(3) $T_n^*(t^2; x) = \left(\frac{n}{n+\beta}\right)^2 \left(x^2 + 2\left(1 + \mu \frac{e_\mu(-nx)}{e_\mu(nx)}\right) \frac{x}{n} + \frac{1}{3n^2}\right) + \frac{2n\alpha}{(n+\beta)^2} \left(x + \frac{1}{2n}\right) + \left(\frac{\alpha}{n+\beta}\right)^2$.

Previous studies demonstrate that providing a better error estimation for positive linear operators plays an important role in approximation theory, which allows us to approximate much faster to the function being approximated. In [5, 6, 7], various better approximation properties of the Szász-Mirakjan-Kantorovich operators, Szász-Mirakjan operators and Szász-Mirakjan-Beta operators were investigated. Recently, in [5, 9], by modifying the Dunkl generalization of Szász-Mirakjan operators, we have showed that our modified operators have better error estimation than the classical ones. In this paper, we apply the Dunkl generalization to the modified Szász-Mirakjan-Kantorovich operators [5] and a better approximation to Szász-Mirakjan-Stancu operators [9].

2. CONSTRUCTION OF OPERATORS

We modify the Szász-Mirakjan-Kantorovich operators [5] and define a Dunkl generalization of these modified operators (or a modification of Dunkl generalization of Szász-Mirakjan-Kantorovich operators[9]) as follows:

Let $\{r_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leq r_n(x) < \infty$ such that

$$r_n(x) = x - \frac{1}{2n}, \quad x \geq \frac{1}{2} \text{ and } n \in \mathbb{N}. \quad (2.1)$$

Then for any $x \geq \frac{1}{2}$, $\mu \geq 0$ and $n \in \mathbb{N}$ we define

$$K_n(f; x) = \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} f(t) dt, \quad (2.2)$$

where $e_\mu(x)$, γ_μ are defined in [16] by (1.4), (1.5) and $f \in C_\zeta[0, \infty)$ with $\zeta \geq 0$ and

$$C_\zeta[0, \infty) = \{f \in C[0, \infty) : |f(t)| \leq M(1+t)^\zeta, \text{ for some } M > 0, \zeta > 0\}. \quad (2.3)$$

If we take $\mu = 0$ in the operator K_n defined by (2.2), then the operator K_n reduces to the modified Szász-Mirakjan-Kantorovich operators given by Oktay Duman et al., [5].

Lemma 2.1. *Let $K_n(\cdot; \cdot)$ be the operators given by (2.2). Then for each $x \geq \frac{1}{2}$, we have we have the following identities:*

- (1) $K_n(1; x) = 1$,
- (2) $K_n(t; x) = x$,
- (3) $K_n(t^2; x) = x^2 + \frac{1}{n} \left(1 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right) x - \frac{1}{n^2} \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)$.

Here we define a modified Szász-Mirakjan-Kantorovich Stancu operators [5] and obtain better approximation results by their Dunkl generalization as follows: For $x \geq \frac{1}{2}$, $\zeta \geq 0$, $n \in \mathbb{N}$, if $f \in C_\zeta[0, \infty)$ satisfying (2.3) with $\zeta \geq 0$, then we define

$$K_n^*(f; x) = \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} f\left(\frac{nt + \alpha}{n + \beta}\right) dt, \quad (2.4)$$

where $e_\mu(x)$, γ_μ are defined in [16] by (1.4), (1.5).

If we take $\alpha = \beta = 0$ in the operator K_n^* defined by (2.4), then the operator K_n^* reduces to operators defined by (2.2). And if we take $\mu = 0$, then it reduces to the operators defined in [5].

Lemma 2.2. *Let $K_n^*(\cdot; \cdot)$ be the operators given by (2.4). Then for each $x \geq \frac{1}{2}$, we have we have the following identities:*

- (1) $K_n^*(1; x) = 1$,
- (2) $K_n^*(t; x) = \frac{n}{n+\beta}x + \frac{\alpha}{n+\beta}$,
- (3) $K_n^*(t^2; x) = \left(\frac{n}{n+\beta}\right)^2 x^2 + \frac{n}{(n+\beta)^2} \left(1 + 2\alpha + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right) x + \frac{1}{(n+\beta)^2} \left(\alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)\right)$.

Lemma 2.3. *Let the operators $K_n^*(\cdot; \cdot)$ be given by (2.4). Then for each $x \geq \frac{1}{2}$, we have*

$$\begin{aligned}
(1) \quad & K_n^*(t-x; x) = \left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha}{n+\beta}, \\
(2) \quad & K_n^*((t-x)^2; x) = \frac{1}{(n+\beta)^2} \left\{ \beta^2 x^2 + \left(n - 2\alpha\beta + 2n\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) x + \alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right\}.
\end{aligned}$$

3. MAIN RESULTS

We obtain the Korovkin's type approximation properties for our operators defined by (2.2).

Let $C_B(\mathbb{R}^+)$ be the set of all bounded and continuous functions on $\mathbb{R}^+ = [0, \infty)$, which is linear normed space with

$$\|f\|_{C_B} = \sup_{x \geq 0} |f(x)|.$$

Let

$$H := \{f : x \in [0, \infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \rightarrow \infty\}.$$

Theorem 3.1. *Let $K_n^*(t; x)$ be the operators defined by (2.4). Then for any function $f \in C_\zeta[0, \infty) \cap H$, $\zeta \geq 2$,*

$$\lim_{n \rightarrow \infty} K_n^*(f; x) = f(x)$$

is uniformly on each compact subset of $[0, \infty)$, where $x \in [\frac{1}{2}, b)$, $b > \frac{1}{2}$.

Proof. The proof is based on Lemma 2.4 and well known Korovkin's theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

$$\lim_{n \rightarrow \infty} K_n^*((t^j; x) = x^j, \quad j = 0, 1, 2, \quad \{\text{as } n \rightarrow \infty\}$$

uniformly on $[0, 1]$.

Clearly $\frac{1}{n} \rightarrow 0$ ($n \rightarrow \infty$) we have

$$\lim_{n \rightarrow \infty} K_n^*(t; x) = x, \quad \lim_{n \rightarrow \infty} K_n^*(t^2; x) = x^2.$$

Which complete the proof. □

We recall the weighted spaces of the functions on \mathbb{R}^+ , which are defined as follows:

$$\begin{aligned}
P_\rho(\mathbb{R}^+) &= \{f : |f(x)| \leq M_f \rho(x)\}, \\
Q_\rho(\mathbb{R}^+) &= \{f : f \in P_\rho(\mathbb{R}^+) \cap C[0, \infty)\}, \\
Q_\rho^k(\mathbb{R}^+) &= \left\{ f : f \in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{x \rightarrow \infty} \frac{f(x)}{\rho(x)} = k (k \text{ is a constant}) \right\},
\end{aligned}$$

where $\rho(x) = 1 + x^2$ is a weight function and M_f is a constant depending only on f . Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm $\|f\|_\rho = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)}$.

Theorem 3.2. Let $K_n^*(t; x)$ be the operators defined by (2.4) acting from $Q_\rho(\mathbb{R}^+) \rightarrow P_\rho(\mathbb{R}^+)$ and satisfying the condition

$$\lim_{n \rightarrow \infty} \| K_n^*(\rho^\tau) - \rho^\tau \|_\varphi = 0, \quad \tau = 0, 1, 2.$$

Then for any function $f \in Q_\rho^k(\mathbb{R}^+)$, we have

$$\lim_{n \rightarrow \infty} \| K_n^*(f; x) - f \|_\varphi = 0.$$

Proof. From [8], we can easily lead to desired result. \square

Theorem 3.3. Let $K_n^*(t; x)$ be the operators defined by (2.4). Then for each function $f \in Q_\rho^k(\mathbb{R}^+)$ we have

$$\lim_{n \rightarrow \infty} \| K_n^*(f; x) - f \|_\rho = 0.$$

Proof. From Lemma 2.2 and Theorem 3.2 for $\tau = 0$, the first condition is fulfilled. Therefore

$$\lim_{n \rightarrow \infty} \| K_n^*(1; x) - 1 \|_\rho = 0.$$

Similarly From Lemma 2.2 and Theorem 3.2 for $\tau = 1, 2$ we have that

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|K_n^*(t; x) - x|}{1 + x^2} &\leq \left| \frac{n}{n + \beta} - 1 \right| \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{\alpha}{n + \beta} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &= \frac{\beta}{2(n + \beta)} + \frac{\alpha}{n + \beta}, \end{aligned}$$

which imply that

$$\lim_{n \rightarrow \infty} \| K_n^*(t; x) - x \|_\rho = 0.$$

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|K_n^*(t^2; x) - x^2|}{1 + x^2} &\leq \frac{2n\beta + \beta^2}{(n + \beta)^2} \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} \\ &\quad + \frac{n}{(n + \beta)^2} \left(1 + 2\alpha + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \\ &\quad + \frac{1}{(n + \beta)^2} \left(\alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right) \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &= \frac{1}{(n + \beta)^2} \left\{ \beta^2 + 2n\beta + \left(\frac{1}{2} + \alpha + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) n \right\} \\ &\quad + \frac{1}{(n + \beta)^2} \left\{ \alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right\}, \end{aligned}$$

which imply that

$$\lim_{n \rightarrow \infty} \| K_n^*(t^2; x) - x^2 \|_\rho = 0.$$

This complete the proof. \square

4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.2) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C_B[0, \infty]$, the space of all bounded and continuous functions on $[0, \infty)$ and $x \geq \frac{1}{2}$. Then for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f, \delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by

$$\omega(f, \delta) = \sup_{|t-x| \leq \delta} |f(t) - f(x)|, \quad t \in [0, \infty). \quad (4.1)$$

It is known that $\lim_{\delta \rightarrow 0+} \omega(f, \delta) = 0$ for $f \in C_B[0, \infty)$ and for any $\delta > 0$ one has

$$|f(t) - f(x)| \leq \left(\frac{|t-x|}{\delta} + 1 \right) \omega(f, \delta). \quad (4.2)$$

Theorem 4.1. *Let $K_n(\cdot; \cdot)$ be the operators defined by (2.2). Then for $f \in C_B[0, \infty)$, $x \geq \frac{1}{2}$ and $n \in \mathbb{N}$ we have*

$$|K_n(f; x) - f(x)| \leq 2\omega(f; \delta_{n,x}),$$

where from Lemma 2.2 we have

$$\delta_{n,x} = \sqrt{K_n((t-x)^2; x)} = \sqrt{\left(1 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right) \frac{x}{n} - \frac{1}{n^2} \left(\frac{5}{12} + \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))}\right)}.$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality.

$$\begin{aligned} & |K_n(f; x) - f(x)| \\ & \leq \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} |f(t) - f(x)| dt \\ & \leq \left\{ \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} \left(1 + \frac{1}{\delta} |t-x|\right) dt \right\} \omega(f; \delta) \\ & = \left\{ 1 + \frac{1}{\delta} \left(\frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} |t-x| dt \right) \right\} \omega(f; \delta) \\ & \leq \left\{ 1 + \frac{1}{\delta} \left(\frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} (t-x)^2 dt \right)^{\frac{1}{2}} (K_n^*(1; x))^{\frac{1}{2}} \right\} \omega(f; \delta) \\ & = \left\{ 1 + \frac{1}{\delta} (K_n(t-x)^2; x)^{\frac{1}{2}} \right\} \omega(f; \delta) \end{aligned}$$

if we choose $\delta = \delta_{n,x}$, the proof is complete. \square

Theorem 4.2. *Let $K_n^*(\cdot; \cdot)$ be the operators defined by (2.4). Then for $f \in C_B[0, \infty)$, $x \geq \frac{1}{2}$ and $n \in \mathbb{N}$ we have*

$$|K_n^*(f; x) - f(x)| \leq 2\omega(f; \delta_{n,x}^*),$$

where $C_B[0, \infty)$ is the space of uniformly continuous bounded functions on \mathbb{R}^+ , $\omega(f, \delta)$ is the modulus of continuity of the function $f \in C_B[0, \infty)$ defined in (4.1) and

$$\delta_{n,x}^* = \sqrt{\frac{1}{(n+\beta)^2} \left\{ \beta^2 x^2 + \left(n - 2\alpha\beta + 2n\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) x + \alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right\}}. \quad (4.3)$$

Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality we can easily get

$$|K_n^*(f; x) - f(x)| \leq \left\{ 1 + \frac{1}{\delta} (K_n^*(t-x)^2; x)^{\frac{1}{2}} \right\} \omega(f; \delta^*)$$

if we choose $\delta^* = \delta_{n,x}^*$ and by applying the result (2) of Lemma 2.3 complete the proof. \square

Remark 4.3. For the operators $T_n^*(\cdot; \cdot)$ defined by (1.6) we may write that, for every $f \in C_B[0, \infty)$, $x \geq 0$ and $n \in \mathbb{N}$

$$|T_n^*(f; x) - f(x)| \leq 2\omega(f; \lambda_{n,x}), \quad (4.4)$$

where from Lemma 1.1 we have

$$\lambda_{n,x} = \sqrt{T_n^*((t-x)^2; x)} = \frac{1}{(n+\beta)^2} \left\{ \beta^2 x^2 + \left(n - \beta(2\alpha+1) + 2n\mu \frac{e_\mu(-nx)}{e_\mu(nx)} \right) x + \alpha^2 + \alpha + \frac{1}{3} \right\}. \quad (4.5)$$

Now we claim that the error estimation in Theorem 4.2 is better than that of (4.4) provided $f \in C_B[0, \infty)$ and $x \geq \frac{1}{2}$. Indeed, for $x \geq \frac{1}{2}$, $\mu \geq 0$ and $n \in \mathbb{N}$, it is guarantees that

$$K_n^*((t-x)^2; x) \leq T_n^*((t-x)^2; x), \quad (4.6)$$

where $K_n^*((t-x)^2; x)$ and $T_n^*((t-x)^2; x)$ are defined in Lemma 2.3 and in (4.5). If we put $\alpha = \beta = 0$ then clearly

$$\left(1 + 2\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \frac{x}{n} - \frac{1}{n^2} \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \leq \left(1 + 2\mu \frac{e_\mu(-nx)}{e_\mu(nx)} \right) \frac{x}{n} + \frac{1}{3n^2}. \quad (4.7)$$

Again if we put $\mu = 0$, then the result in [5] by equation (3.6) is obtained as

$$\frac{x}{n} - \frac{5}{12n^2} \leq \frac{x}{n} + \frac{1}{3n^2}. \quad (4.8)$$

Now we give the rate of convergence of the operators $K_n^*(f; x)$ defined in (2.4) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$.

Let $f \in C_B[0, \infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f : |f(\zeta_1) - f(\zeta_2)| \leq M |\zeta_1 - \zeta_2|^\nu \quad (\zeta_1, \zeta_2 \in [0, \infty))\} \quad (4.9)$$

Theorem 4.4. Let $K_n^*(\cdot; \cdot)$ be the operator defined in (2.4). Then for each $f \in Lip_M(\nu)$, ($M > 0$, $0 < \nu \leq 1$) satisfying (4.9) we have

$$|K_n^*(f; x) - f(x)| \leq M (\delta_{n,x}^*)^{\frac{\nu}{2}}$$

where $\delta_{n,x}^*$ is given in Theorem 4.3.

Proof. We prove it by using (4.9) and Hölder inequality.

$$\begin{aligned} |K_n^*(f; x) - f(x)| &\leq |K_n^*(f(t) - f(x); x)| \\ &\leq K_n^*(|f(t) - f(x)|; x) \\ &\leq |MK_n^*(|t - x|^\nu; x)|. \end{aligned}$$

Therefore

$$\begin{aligned} &|K_n^*(f; x) - f(x)| \\ &\leq M \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} |t - x|^\nu dt \\ &\leq M \frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \left(\frac{(nr_n(x))^k}{\gamma_\mu(k)} \right)^{\frac{2-\nu}{2}} \\ &\quad \times \left(\frac{(nr_n(x))^k}{\gamma_\mu(k)} \right)^{\frac{\nu}{2}} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} |t - x|^\nu dt \\ &\leq M \left(\frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} dt \right)^{\frac{2-\nu}{2}} \\ &\quad \times \left(\frac{n}{e_\mu(nr_n(x))} \sum_{k=0}^{\infty} \frac{(nr_n(x))^k}{\gamma_\mu(k)} \int_{\frac{k+2\mu\theta_k}{n}}^{\frac{k+1+2\mu\theta_k}{n}} |t - x|^2 dt \right)^{\frac{\nu}{2}} \\ &= M (K_n^*(t - x)^2; x)^{\frac{\nu}{2}}. \end{aligned}$$

Which complete the proof. \square

Let $C_B[0, \infty)$ denote the space of all bounded and continuous functions on $\mathbb{R}^+ = [0, \infty)$ and

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\}, \quad (4.10)$$

with the norm

$$\|g\|_{C_B^2(\mathbb{R}^+)} = \|g\|_{C_B(\mathbb{R}^+)} + \|g'\|_{C_B(\mathbb{R}^+)} + \|g''\|_{C_B(\mathbb{R}^+)}, \quad (4.11)$$

also

$$\|g\|_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} |g(x)|. \quad (4.12)$$

Theorem 4.5. Let $K_n^*(\cdot; \cdot)$ be the operator defined in (2.4). Then for any $g \in C_B^2(\mathbb{R}^+)$ we have

$$|K_n^*(f; x) - f(x)| \leq \left\{ \left(\left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha}{n+\beta} \right) + \frac{\delta_{n,x}^*}{2} \right\} \|g\|_{C_B^2(\mathbb{R}^+)},$$

where $\delta_{n,x}^*$ is given in Theorem 4.3.

Proof. Let $g \in C_B^2(\mathbb{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi) \frac{(t-x)^2}{2}, \quad \psi \in (x, t).$$

By applying linearity property on K_n^* , we have

$$K_n^*(g, x) - g(x) = g'(x)K_n^*((t-x); x) + \frac{g''(\psi)}{2}K_n^*((t-x)^2; x),$$

which imply that

$$\begin{aligned} |K_n^*(g; x) - g(x)| &\leq \left(\left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha}{n+\beta} \right) \|g'\|_{C_B(\mathbb{R}^+)} \\ &\quad + \left\{ \frac{1}{(n+\beta)^2} \left(\beta^2 x^2 + \left(n - 2\alpha\beta + 2n\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) x + \alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right) \right\} \frac{\|g''\|_{C_B(\mathbb{R}^+)}}{2}. \end{aligned}$$

From (4.11) we have $\|g'\|_{C_B[0, \infty)} \leq \|g\|_{C_B^2[0, \infty)}$.

$$\begin{aligned} |K_n^*(g; x) - g(x)| &\leq \left(\left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha}{n+\beta} \right) \|g\|_{C_B^2(\mathbb{R}^+)} \\ &\quad + \left\{ \frac{1}{(n+\beta)^2} \left(\beta^2 x^2 + \left(n - 2\alpha\beta + 2n\mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) x + \alpha^2 - \left(\frac{5}{12} + \mu \frac{e_\mu(-nr_n(x))}{e_\mu(nr_n(x))} \right) \right) \right\} \frac{\|g\|_{C_B^2(\mathbb{R}^+)}}{2}. \end{aligned}$$

This completes the proof from 2 of Lemma 2.3. \square

The Peetre's K -functional is defined by

$$K_2(f, \delta) = \inf_{C_B^2(\mathbb{R}^+)} \left\{ \left(\|f - g\|_{C_B(\mathbb{R}^+)} + \delta \|g''\|_{C_B^2(\mathbb{R}^+)} \right) : g \in \mathcal{W}^2 \right\}, \quad (4.13)$$

where

$$\mathcal{W}^2 = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\}. \quad (4.14)$$

There exists a positive constant $C > 0$ such that $K_2(f, \delta) \leq C\omega_2(f, \delta^{\frac{1}{2}})$, $\delta > 0$, where the second order modulus of continuity is given by

$$\omega_2(f, \delta^{\frac{1}{2}}) = \sup_{0 < h < \delta^{\frac{1}{2}}} \sup_{x \in \mathbb{R}^+} |f(x+2h) - 2f(x+h) + f(x)|. \quad (4.15)$$

Theorem 4.6. Let $K_n^*(\cdot; \cdot)$ be the operator defined in (2.2) and $C_B[0, \infty)$ be the space of all bounded and continuous functions on \mathbb{R}^+ . Then for $x \geq \frac{1}{2}$ and $f \in C_B(\mathbb{R}^+)$ we have

$$|K_n^*(f; x) - f(x)|$$

$$\leq 2M \left\{ \omega_2 \left(f; \sqrt{\frac{(\frac{2n}{n+\beta}-2)x + \frac{2\alpha}{n+\beta} + \delta_{n,x}^*}{4}} \right) + \min \left(1, \frac{(\frac{2n}{n+\beta}-2)x + \frac{2\alpha}{n+\beta} + \delta_{n,x}^*}{4} \right) \|f\|_{C_B(\mathbb{R}^+)} \right\},$$

where M is a positive constant, $\delta_{n,x}^*$ is given in Theorem 4.3 and $\omega_2(f; \delta)$ is the second order modulus of continuity of the function f defined in (4.15).

Proof. We prove this by using the Theorem (4.5)

$$\begin{aligned} |K_n^*(f; x) - f(x)| &\leq |K_n^*(f - g; x)| + |K_n^*(g; x) - g(x)| + |f(x) - g(x)| \\ &\leq 2 \|f - g\|_{C_B(\mathbb{R}^+)} + \frac{\delta_{n,x}^*}{2} \|g\|_{C_B^2(\mathbb{R}^+)} \\ &\quad + \left(\left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha}{n+\beta} \right) \|g\|_{C_B(\mathbb{R}^+)} \end{aligned}$$

From (4.11) clearly we have $\|g\|_{C_B[0,\infty)} \leq \|g\|_{C_B^2[0,\infty)}$.

Therefore,

$$|K_n^*(f; x) - f(x)| \leq 2 \left(\|f - g\|_{C_B(\mathbb{R}^+)} + \frac{\left(\frac{2n}{n+\beta} - 2 \right) x + \frac{2\alpha}{n+\beta} + \delta_{n,x}^*}{4} \|g\|_{C_B^2(\mathbb{R}^+)} \right),$$

where $\delta_{n,x}^*$ is given in Theorem 4.3.

By taking infimum over all $g \in C_B^2(\mathbb{R}^+)$ and by using (4.13), we get

$$|K_n^*(f; x) - f(x)| \leq 2K_2 \left(f; \frac{\left(\frac{2n}{n+\beta} - 2 \right) x + \frac{2\alpha}{n+\beta} + \delta_{n,x}^*}{4} \right)$$

Now for an absolute constant $D > 0$ in [4] we use the relation

$$K_2(f; \delta) \leq D\{\omega_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|\}.$$

This complete the proof. \square

5. CONCLUDING REMARKS AND OBSERVATIONS

Motivated essentially by the recent investigation of dunkl type generalization of Stancu operators by Gürhan İçöz [9] and modified Szász-Mirakjan-Kantrovich operators by Oktay Duman [5], we have proved several approximation results. The operators defined in (2.4), modifying the recent investigation of Gürhan İçöz [9] and also if we take $\alpha = \beta = 0$ in operators defined in (2.4), then it reduces to the operators (2.2), which becomes a dunkl generalization of the paper investigated by Oktay Duman [5]. We have successfully extended these results and modifying the results of papers ([9, 5]). Several other related results have also been considered.

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